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GREEN'S FUNCTION APPROACH TO THE NONLINEAR BENDING OF CLOSELY-SPACED PARALLEL PLATES

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Abstract—This article presents a new extension of the Green's function method in computational mechanics. An iterative procedure is developed for analyzing the contact interaction in a system of closely-spaced parallel thin plates, possibly situated just above a Winkler foundation. The plates have uniform thicknesses and are composed of isotropic homogeneous elastic materials. Frictionless contact is also assumed. According to the classification proposed by Dundurs and Stippes (1970), the *advancing* contact has been discussed. The formulation of the problem brings a combination of two types of nonlinearities of different origins. The geometric nonlinearity resulting from relatively large deflections of the plates is accompanied by a nonlinearity which is due to the fact that the boundaries between contact and non-contact zones for each pair of plates are initially unknown. Linear problems appearing within each iteration are attacked by a version of the Green's function method. The technique utilizes the analytically constructed Green's functions and matrices for the biharmonic equation and Lame's system of the plane problem in the theory of elasticity. Contact conditions for each plate in the system are treated by implementing penalty functions. Numerical results are encouraging, and an extension of this study to more complicated formulations is currently under way.

INTRODUCTION

A number of analytical methods based on Green's functions have originated from Dniepropetrovsk State University, Ukraine, within the past two decades. Special techniques for constructing Green's functions and matrices for boundary value problems in partial differential equations and systems of the elliptic type have been presented by Melnikov (1977a) and developed in a series of works by Dolgova and Melnikov (1978, 1989); Davydov *et al.* (1978); and Melnikov (1981, 1991). In addition, a variety of extremely complex problems in applied mechanics have already been attacked by the Green's function method. They include the following: heat conduction (Melnikov, 1970, 1976, 1977a, 1982; Melnikov and Dolgova, 1976); elastic torsion (Melnikov, 1977a, 1982); the plane problem in the theory of elasticity (Melnikov, 1977a,b, 1982; Dolgova and Melnikov, 1978, 1989; Koshnarjova and Melnikov, 1991a,b; Melnikov and Koshnarjova, 1994); linear and geometrically nonlinear problems in the theory of plates and shells (Melnikov and Tsadikova, 1978; Melnikov and Bajrak, 1980; Melnikov, 1982; Melnikov and Voloshko, 1988; Shubenko, 1990; Melnikov and Shubenko, 1993); contact mechanics with initially unknown contact zones (Koshnarjova and Melnikov, 1991a,b; Koshnarjova *et al.*, 1987); and shape optimization in the theory of elasticity (Melnikov, 1982; Melnikov and Titarenko, 1992, 1993). Some theoretical and computational aspects of the Green's function method have been studied by Melnikov (1970, 1981, 1982, 1985, 1991) and Koshnarjova and Melnikov (1986, 1991b).

The present study is a continuation of this extensive list of works based on the Green's function method. Herein, the method is extended to a new class of boundary value problems of applied mechanics. An iterative procedure has been developed to compute the characteristics of the elastic equilibrium of a system of parallel thin rectangular plates located just above a Winkler foundation. The analysis is executed within the framework of a geometrically nonlinear (Karman, 1910) formulation of the problem. Each single iteration utilizes the appropriate Green's function for the biharmonic equation and Green's matrix for Lame's system of the displacement formulation of the plane problem in the theory of elasticity. The technique proposed by Melnikov (1977a) is used to analytically construct the Green's functions and matrices needed. Contact conditions for the deflection function for each plate are treated by the penalty functions method. This approach to the problem provides a fast and stable convergence of the iterative process, resulting in a final solution that is acceptably accurate.

Starting from the early 1930s, considerable effort has been put forth in the investigation of the contact problems in the theory of plates. In the earlier studies by Girkmann (1931), and Hofmann (1938), a class of problems is discussed involving circular plates, portions of which are constrained from deflection by the presence of an absolutely rigid plane surface parallel to the middle plane of the plate. These studies have formulated the contact problems within the scope of the classical Poisson–Kirchhoff plate theory. Later it was shown that, as far as contact problems are concerned, the classical plate theory cannot adequately describe the stress–strain state of the contacting portions of plates. This is especially true either if the plate is interacting with an absolutely rigid constraining surface, or if the plates are in partial contact. Eliminating the aforementioned disadvantage, Naghdi and Rowley (1953), Frederick (1956) and Essenburg (1962) have stated problems of the class mentioned above within the scope of the Reissner plate theory which accounts for the effect of the transverse normal stress and transverse shear deformation. A variety of other formulations of contact problems in the theory of plates and shells have recently been considered. (See, for instance, Weitsman, 1969; Grigolyuk and Tolkachev, 1980; Dampsey *et al.*, 1984; Ascione and Olivito, 1985; Rajapakse and Salvadurai, 1986.)

Two points clearly accentuate the novelty in the present work. First, this study broadens the range of applications of the Green's function method in computational mechanics. Secondly, the contact interaction of the system of parallel plates has, for the first time, been considered within the scope of the geometrically nonlinear Karman's formulation.

PROBLEM FORMULATION. THE ITERATIVE PROCEDURE

Consider a system of N parallel thin elastic plates, enumerated upward from the bottom, occupying a region Ω bounded with a piece-wise smooth contour Γ, as shown in Fig. 1. Let a transverse loading $q(x,y)$ be applied to the N th plate. Set the lowest of the plates (plate 1) just above a Winkler foundation. Assume that each plate has a uniform thickness $h_n (n = 1, \ldots, N)$, and is composed of a homogeneous isotropic material whose elastic modulus and Poisson ratio are E_n and ν_n respectively. Denote with δ_n the initial distance between the middle planes of the $(n-1)$th and nth plates $(n = 2, \ldots, N)$, with δ_1 denoting the initial distance between the bottom face of the lowest plate and the foundation. In addition, assume a frictionless contact of the plates in the system. Assume also that the values of the deflections of each plate in the system are commensurate with its thickness. Suppose that the equilibrium state of the nth plate can be adequately described by the displacement formulation proposed by Karman (1910):

$$D_n \cdot \nabla^2 \nabla^2 w_n = q_n + S(u_n, v_n, w_n),$$
$$L[U_n] = P(w_n), \quad n = 1, \ldots, N, \tag{1}$$

where D_n is the bending rigidity of the nth plate, $w_n = w_n(x,y)$ is the deflection function of the nth plate, $q_n = q_n(x,y)$ represents a transverse loading applied to this plate as a result of its contact interaction with the $(n-1)$th and $(n+1)$th plates, U_n is a vector of in-plane displacements of the nth plate, whose components are u_n and v_n. The nonlinear operator $S(u, v, w)$ is given by

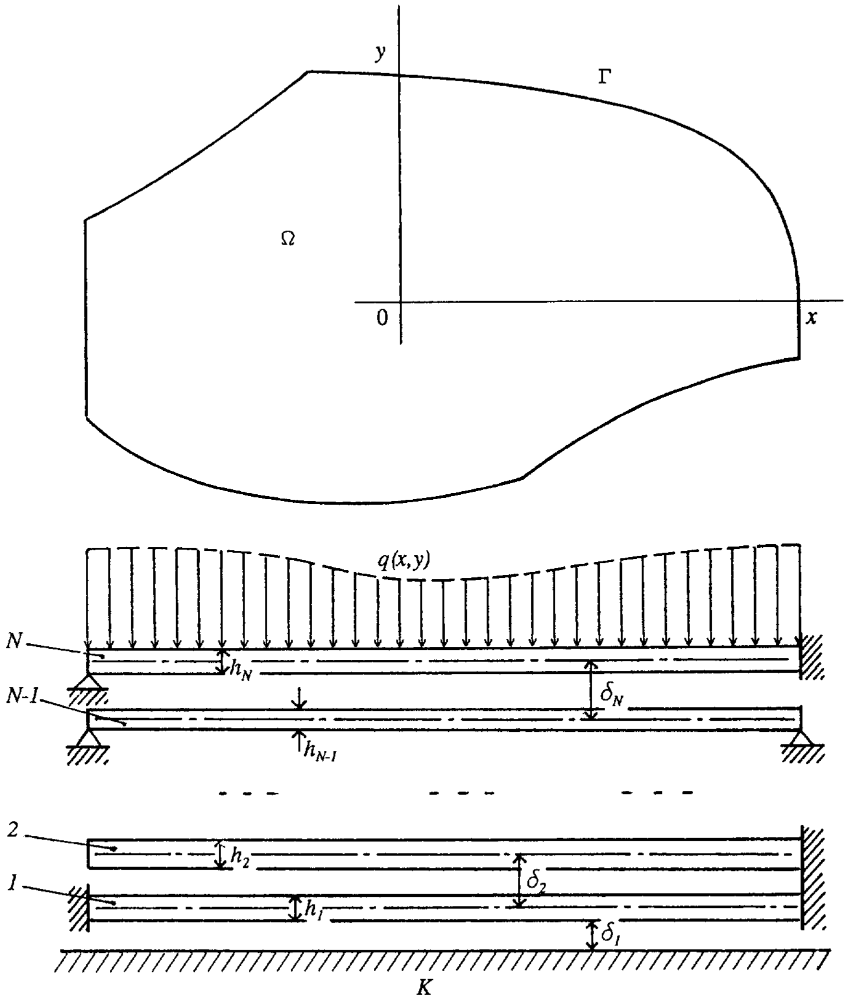

Fig. 1. The configuration of the plates and operative coordinate system.

$$S(u,v,w) \equiv (Eh/(1-\nu^2)) \cdot \{[\partial u/\partial x + 1/2 \cdot (\partial w/\partial x)^2 + \nu(\partial v/\partial y + 1/2 \cdot (\partial w/\partial y)^2)] \cdot \partial^2 w/\partial x^2$$
$$+ [\nu(\partial u/\partial x + 1/2 \cdot (\partial w/\partial x)^2) + \partial v/\partial y + 1/2 \cdot (\partial w/\partial y)^2] \cdot \partial^2 w/\partial y^2$$
$$+ (1-\nu) \cdot (\partial u/\partial y + \partial v/\partial x + \partial w/\partial x \cdot \partial w/\partial y) \cdot \partial^2 w/\partial x \partial y\}.$$

$L[U_n]$ is Lame's operator of the plane problem in the theory of elasticity, while $P(w)$ represents a nonlinear vector operator whose components $P_1(w)$ and $P_2(w)$ are expressed as follows:

$$P_1(w) \equiv -2\partial w/\partial x \cdot (\partial^2 w/\partial x^2 + (1-\nu)\partial^2 w/\partial y^2) - (1+\nu)\partial w/\partial y \cdot \partial^2 w/\partial x\,\partial y,$$
$$P_2(w) \equiv -2\partial w/\partial y \cdot (\partial^2 w/\partial y^2 + (1-\nu)\partial^2 w/\partial x^2) - (1+\nu)\partial w/\partial x \cdot \partial^2 w/\partial x\,\partial y.$$

Boundary conditions on Γ for each plate in the system are specified in the form

$$B_1^n(w_n) = 0\,, \quad B_2^n(w_n) = 0\,, \quad (x,y) \in \Gamma \tag{2}$$

$$C_1^n(u_n, v_n) = 0\,, \quad C_2^n(u_n, v_n) = 0\,, \quad (x,y) \in \Gamma. \tag{3}$$

These equations are a generalization of the most frequently encountered boundary conditions (simply supported, clamped, or free edge). In addition, the deflection function $w_n(x, y)$ is assumed to satisfy the linearized contact conditions written in the form

$$w_n(x,y) - w_{n-1}(x,y) - \delta_n \leqslant 0, \quad (n = 2, \ldots, N) \tag{4}$$

$$w_1(x,y) - \delta_1 \leqslant 0. \tag{5}$$

As shown by Koltunov *et al.* (1983), the conditions above can be considered as the first asymptotic approximation to the corresponding nonlinear conditions.

Hence, the geometrically nonlinear bending of the system of N parallel plates is described by the system of equations [eqn (1)] along with the boundary and contact conditions in eqns (2)–(5). As we have already mentioned, this formulation implies two different types of nonlinearities. Before we treat the formulation in full terms, let us concentrate on the nonlinearity which is due to the fact that the boundary between contact and noncontact zones is initially unknown. In doing that, we first restrict our consideration to a system of biharmonic equations:

$$D_n \cdot \nabla^2 \nabla^2 w_n - q_n = 0, \quad (n = 1, \ldots, N), \tag{6}$$

subject to the boundary conditions in eqn (2) along with the contact conditions in eqns (4) and (5). To analyze the problem, we introduce Sobolev's spaces $H_{n1} \equiv W_2^2(\Omega)$ of functions $w_n(x, y)$ which are summable over Ω along with their squares and second-order partial derivatives. Construct a direct product $H_1 = H_{11} \otimes H_{21} \otimes \ldots \otimes H_{N1}$ and define a set $T_1 \subset H_1$

$$\begin{aligned} T_1 = \{ W = \{w_1, w_2, \ldots, w_N\} \in H_1 : \\ & w_n(x,y) - w_{n-1}(x,y) - \delta_n \leqslant 0, \quad \forall (x,y) \in \Omega, \quad (n = 2, \ldots, N), \\ & w_1(x,y) - \delta_1(x,y) \leqslant 0, \quad \forall (x,y) \in \Omega, \\ & B_1^n(w_n)\,|_{(x,y)\in\Gamma} = 0, \quad B_2^n(w_n)\,|_{(x,y)\in\Gamma} = 0, \quad (n = 1, \ldots, N)\} \end{aligned} \tag{7}$$

of kinematically permissible deflections, including the set of the deflections $w_n(x, y)$, $(n = 1, \ldots, N)$ satisfying conditions in eqns (2), (4) and (5). As shown by Duvant and Lions (1972), and later by Panagiotopoulos (1985), a solution of the problem in eqns (2) and (4)–(6) should also satisfy the variational inequality given by

$$Q_1(W, W^* - W) \geqslant L_1(W^* - W), \quad \forall W^* \in T_1, \tag{8}$$

where $Q_1(W, W^*)$ and $L_1(W^*)$ are quadratic and linear forms, respectively, defined by

$$\begin{aligned} Q_1(W, W^*) \equiv \sum_{n=1}^{N} D_n \iint_\Omega [\partial^2 w_n/\partial x^2 \cdot \partial^2 w_n^*/\partial x^2 + \partial^2 w_n/\partial y^2 \cdot \partial^2 w_n^*/\partial y^2 + \nu_n \cdot (\partial^2 w_n/\partial x^2 \\ \times \partial^2 w_n^*/\partial y^2 + \partial^2 w_n/\partial y^2 \cdot \partial^2 w_n^*/\partial x^2) + 2(1-\nu_n) \cdot \partial^2 w_n/\partial x\,\partial y \cdot \partial^2 w_n^*/\partial x\,\partial y]\,\mathrm{d}\Omega, \end{aligned} \tag{9}$$

$$L_1(W^*) \equiv \iint_\Omega q \cdot w_N^*\,\mathrm{d}\Omega. \tag{10}$$

Notice that the variational inequality in eqn (8) is not quite appropriate for directly computing the solution of the problem under consideration. It appears in practice, however (Glowinski *et al.*, 1976), that its equivalent formulation, given by

$$\inf_{W^* \varepsilon T_1} \{J(W^*) \equiv 0.5 Q_1(W, W^*) - L_1(W^*)\}, \tag{11}$$

provides a more convenient way for computing. Vasilyev (1981) has analyzed in detail a variety of methods that have been developed for the solution of extremal problems of the type in eqn (11). In this study, we use the penalty functions technique to reduce the problem in eqn (11) to a certain nonlinear boundary value problem.

To describe the computational algorithm based on the method of penalty functions, we introduce a perturbated functional $J_p(W^*)$ as follows:

$$J_p(W^*) = J(W^*) + 0.5\left[\sum_{n=2}^{N} \iint_\Omega \beta_p(w_n^*, w_{n-1}^*)\cdot(w_n^* - w_{n-1}^* - \delta_n)^2\,\mathrm{d}\Omega + \iint_\Omega \gamma_p(w_1^*)\cdot(w_1^* - \delta_1)^2\,\mathrm{d}\Omega\right], \tag{12}$$

defined over a set T_1^* given by

$$T_1^* = \{W = \{w_1, w_2, \ldots, w_N\} \in H: B_1^n(w_n)|_{(x,y)\in\Gamma} = 0, \quad B_2^n(w_n)|_{(x,y)\in\Gamma} = 0, \quad n = 1, \ldots, N\}. \tag{13}$$

Functions β_p and γ_p in eqn (12) are determined by

$$\beta_p(w_n, w_{n-1}) = \begin{cases} 0, & w_n - w_{n-1} - \delta_n \leqslant 0, \\ \varepsilon_1^{-2}, & w_n - w_{n-1} - \delta_n > 0; \end{cases} \tag{14}$$

$$\gamma_p(w_1) = \begin{cases} 0, & w_1 - \delta_1 \leqslant 0, \\ \varepsilon_2^{-2}, & w_1 - \delta_1 > 0. \end{cases} \tag{15}$$

Denote by $W^p(x, y)$ a vector-function providing the minimum of the perturbated functional J_p determined by eqn (12), and therefore representing an approximate solution of the problem in eqn (11). Then inequalities

$$w_n^p - w_{n-1}^p - \delta_n > 0, \quad n = 2, \ldots, N \tag{16}$$

dealing with the components in $W^p(x, y)$ determine the shape of the contact zone for the $(n-1)$th and nth plates, while an inequality of the form

$$w_1^p - \delta_1 > 0 \tag{17}$$

determines a contact zone for the lowest plate and the foundation.

Vasilyev (1981) provides a detailed discussion on convergence for our application of the method of penalty functions. In particular, it has been shown that as $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, the approximate solution W^p converges weakly in H_1 to the exact solution W of the problem in eqn (11).

We know that a system of the Euler's equations matching the functional (Lagrangian) in eqn (12) can be written in the form:

$$D_1 \cdot \nabla^2\nabla^2 w_1^p - F_1(w_1^p, w_2^p) = 0, \tag{18}$$

$$D_n \cdot \nabla^2\nabla^2 w_n^p - F_n(w_{n-1}^p, w_n^p, w_{n+1}^p) = 0, \quad n = 2, \ldots, N-1, \tag{19}$$

$$D_N \cdot \nabla^2\nabla^2 w_N^p - q - F_N(w_{N-1}^p, w_N^p) = 0, \tag{20}$$

where we use the following notation

$$F_1(w_1, w_2) = \beta_p(w_1, w_2)\cdot(w_2 - w_1 - \delta_2) - \gamma_p(w_1)\cdot(w_1 - \delta_1),$$

$$F_n(w_{n-1}, w_n, w_{n+1}) = \beta_p(w_{n+1}, w_n)\cdot(w_{n+1} - w_n - \delta_{n+1})$$

$$-\beta_p(w_n, w_{n-1})\cdot(w_n - w_{n-1} - \delta_n), \quad n = 2,\dots,N-1,$$

$$F_N(w_{N-1}, w_N) = -\beta_p(w_N, w_{N-1})\cdot(w_N - w_{N-1} - \delta_N).$$

Thus, the approximate solution W^p of the problem in eqn (11) is supposed to satisfy the system of nonlinear equations in eqns (18)–(20) along with the boundary conditions in eqn (2). Notice the importance of finding the values of ε_1 and ε_2 which provide a sufficiently accurate approximate solution W^p. *A priori* estimation made by Glowinski *et al.* (1976) indicates that the value of max $\{\varepsilon_1, \varepsilon_2\}$ should be minimized to give us the most accurate result possible. However, small values of ε_1 and ε_2 cause excessive nonlinearity in the problem. To overcome this predicament, we reformulate the conditions in eqns (4) and (5) as follows

$$w_n^p - w_{n-1}^p - \delta_n \leqslant \sigma_2 + \theta_2 \cdot \max_{(x,y)\in\Omega} |w_n^p|, \quad n = 2,\dots,N-1, \tag{21}$$

$$w_1^p - \delta_1 \leqslant \sigma_1 + \theta_1 \cdot \max_{(x,y)\in\Omega} |w_1^p|, \tag{22}$$

where σ_2 and θ_2 are permissible values of absolute and relative errors, respectively.

Notice that the differential equation [eqn (18)] could be applied to the case in which the bottom surface of the lowest plate interacts with an elastic foundation whose Winkler coefficient is ε_2^{-2}. Therefore, if the system of parallel plates is situated just above a Winkler foundation, then the quantity ε_2^{-2} in eqn (15) must be replaced with the actual value of the Winkler coefficient.

GREEN'S FUNCTION APPROACH

To compute a numerical solution to the nonlinear boundary value problem in eqns (2) and (18)–(20), we implement an iterative scheme that is specially adapted to the corresponding Green's functions for the biharmonic equation. According to that scheme, the approximate value $w_n^p(x, y)$ of the deflection function $w_n(x, y)$, $(n = 1,\dots,N)$ for each plate in the system is considered as a limit of the functional sequence $\{(w_n^p(x, y))_k\}$, $(k = 0, 1, 2, \dots)$ arising from a system of linear boundary value problems as written:

$$D_1 \cdot \nabla^2\nabla^2 (w_1^p)_{k+1} = (1-\tau_{k+1})\cdot D_1 \cdot \nabla^2\nabla^2 (w_1^p)_k + \tau_{k+1}\cdot F_1((w_1^p)_k, (w_2^p)_k), \tag{23}$$

$$D_n \cdot \nabla^2\nabla^2 (w_n^p)_{k+1} = (1-\tau_{k+1})\cdot D_n \cdot \nabla^2\nabla^2 (w_n^p)_k$$

$$+\tau_{k+1}\cdot F_n((w_{n-1}^p)_k, (w_n^p)_k, (w_{n+1}^p)_k), \quad n = 2,\dots,N-1, \tag{24}$$

$$D_N \cdot \nabla^2\nabla^2 (w_N^p)_{k+1} = (1-\tau_{k+1})\cdot D_N \nabla^2\nabla^2 (w_N^p)_k + \tau_{k+1}\cdot[q + F_N((w_{N-1}^p)_k, (w_N^p)_k)], \tag{25}$$

$$B_1^n[(w_n^p)_{k+1}]|_{(x,y)\in\Gamma} = 0, \quad B_2^n[(w_n^p)_{k+1}]|_{(x,y)\in\Gamma} = 0, \quad n = 1,\dots,N. \tag{26}$$

In the formulation above, the subscript k indicates the current iteration number. The parameter τ_{k+1} could be varied with k, allowing us to properly manage the rate of convergence of the iterative process. In the next section of this paper, we analyze the convergence in detail.

The iterative process in eqns (23)–(26) is called (Samarski and Gulin, 1989) the non-stationary double-layered iterative scheme. Clearly, the particular case associated with $\tau_{k+1} = 1$ represents the simplest version of the iteration method (successive approximations). Initial approximations $(w_n^p)_0$ for the scheme in eqns (23)–(26) can be taken, for instance, as $[w_n^p(x,y)]_0 \equiv 0$, $(n = 1, \ldots, N)$, unless a better suggestion is available.

We will apply the Green's function method to the linear boundary value problems formulated in eqns (23)–(26). The advantage in this method over other techniques, such as the finite element or finite difference methods, is its ability to compute all of the important characteristics of the stress–strain state with an equal level of accuracy. This superiority arises from the fact that, in developing our algorithm, we avoid any procedures of numerical differentiation. Indeed, the Green's function representation enables us to do all of the differentiation analytically. The only numerical procedures that are employed are those used to approximate integrals of the form in eqns (27) and (28).

To determine each single approximation $[w_n^p(x,y)]_{k+1}$ to the solution of the nonlinear problem in eqns (2) and (18)–(20), in compliance with the formulation in eqns (23)–(26), it is necessary to first evaluate the right-hand terms in eqns (23)–(25). These terms are obtained by applying the right-hand side operators in eqn (23)–(25) to the preceding approximation $[w_n^p(x,y)]_k$. Consequently, convergence of the iterative process critically depends on the accuracy of the numerical evaluations within each step.

Let $G_B(x,y;\xi,\zeta)$ represent the Green's function of the biharmonic equation for the region Ω, with the boundary conditions of the form in eqn (2) prescribed along its contour Γ. Then the solution to the nonhomogeneous equation $\nabla^2\nabla^2\, w(x,y) = -F(x,y)$ satisfying these conditions is expressible in terms of the integral

$$w(x,y) = \iint_{\Omega} G_B(x,y;\,\xi,\zeta)\cdot F(\xi,\zeta)\,\mathrm{d}\Omega(\xi,\zeta). \tag{27}$$

Hence, to successfully run each single loop in the iterative process defined by eqns (23)–(26), it is necessary to overcome all computational difficulties arising from the evaluation of the proper as well as improper integrals written in the form

$$\iint_{\Omega} K_B(x,y;\xi,\zeta)\cdot R_B(\xi,\zeta)\,\mathrm{d}\Omega(\xi,\zeta). \tag{28}$$

The kernel function $K_B(x,y;\xi,\zeta)$ in this integral results from the effect of the right-hand side operators in eqns (23)–(25) on the Green's function $G_B(x,y;\xi,\zeta)$, while the factor $R_B(\xi,\zeta)$ in the integrand of eqn (28) is obtained by recomputing the integrals of the form in eqn (28).

We now describe a procedure for approximating the integrals in eqns (27) and (28) for the rectangular region Ω $(0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b)$. For this purpose, we partition the region Ω into a set of elementary rectangles Ω_m, $(m = 1, \ldots, M)$

$$\Omega = \bigcup_{m=1}^{M} \Omega_m, \quad \Omega_m = \{(x,y)\in R^2:\ x_m^1 \leqslant x \leqslant x_m^2,\ y_m^1 \leqslant y \leqslant y_m^2\}.$$

We then use the following cubature formula

$$w_h(x,y) = \sum_{m=1}^{M} F(x_m,y_m)\cdot\int_{x_m^1}^{x_m^2}\int_{y_m^1}^{y_m^2} G_B(x,y;\,\xi,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta, \tag{29}$$

where

$$x_m = 0.5\,(x_m^1+x_m^2), \quad y_m = 0.5\,(y_m^1+y_m^2).$$

Let the function $F(\xi,\zeta)$ satisfy the Lipschitz condition of the form

$$|F(\xi_1,\zeta_1)-F(\xi_2,\zeta_2)| \leqslant \mathscr{L}_m\cdot[(\xi_1-\xi_2)^2+(\zeta_1-\zeta_2)^2]^{1/2} \tag{30}$$

in each of the elementary rectangles Ω_m. Since the Green's function $G_B(x,y;\xi,\zeta)$ is absolutely integrable over Ω for any fixed position of the point (x,y), one can readily obtain an estimation:

$$|w_h(x,y)-w(x,y)| \leqslant \max\{\mathscr{L}_m\cdot \operatorname{diam}\Omega_m\}\cdot \int_0^a\int_0^b G_B(x,y;\xi,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta, \tag{31}$$

of the error of the cubature formula given in eqn (29).

Notice that the specific form of the analytic representations of Green's functions obtained by using the technique proposed by Melnikov (1977a) allows us to evaluate the integrals in eqn (29) analytically. This significantly enhances the accuracy of the numerical procedure.

DETERMINATION OF PARAMETERS IN THE ITERATIVE PROCEDURE

As we have already mentioned, obtaining favorable results for the iterative procedure in eqns (23)–(26) critically depends on selection of appropriate values of the parameter τ_{k+1} at each stage of the procedure to provide a stable and fast convergence of the iterative process. Our previous computations showed that in the stationary ($\tau_{k+1} = 1$) scheme, the value of the uniform right-hand term q in eqn (25) for which the process does actually converge increases as the value of τ_{k+1} decreases. However, in this case, the convergence is hindered. It was also found that if the process diverges, the sequence of the largest differences of the two successive approximations of the deflection function $w(x,y)$ alternates.

To overcome these difficulties, we propose the following heuristical algorithm for choosing the sequence of values τ_{k+1} of the parameter τ for the scheme presented in eqns (23)–(26). First assume that the kth approximation $[w(x,y)]_k$ of the displacement function and an associate value of the parameter τ_k are already available. We then obtain the next two approximations of the displacement function by using the scheme in eqns (23)–(26), with an assumption that $\tau_{k+2} = \tau_{k+1} = \tau_k$. We now denote

$$\lambda_{k+1} = [w(x^*,y^*)]_{k+1}-[w(x^*,y^*)]_k, \tag{32}$$

where (x*,y*) is the point on the middle plane of the plate at which the kth approximation of the deflection function achieves its maximum absolute value. If the following condition

$$\lambda_{k+1}\cdot\lambda_{k+2} \geqslant -\alpha\cdot\lambda_{k+1}^2 \tag{33}$$

holds for $0 < \alpha < 1$ (where α is the coefficient of condensing in the iterative process), then the last two approximations are successful and the iteration can proceed. If the left-hand term of the inequality in eqn (33) is positive, then the value of τ for the continuation of the process may be slightly increased by assuming that $\tau_{k+4} = \tau_{k+3} = \mu_i\cdot\tau_k$. The value of $\mu_i \geqslant 1$ is the coefficient of the step increase in the iterative process. Unless the condition in eqn (33) holds, the evaluation of the $(k+1)$th and $(k+2)$th approximations must be restarted with the assumption that $\tau_{k+1} = \tau_{k+2} = \mu_d\cdot\tau_k$ where $\mu_d < 1$ is the coefficient of decrease of the step in the iterative process.

We had concluded earlier that the condition given by

$$(\tau_{k+2})^{-1}\cdot\max_{\Omega}|w_{k+2}-w_{k-1}| \leqslant \rho+\omega\cdot\max_{\Omega}|w_{k-1}|, \tag{34}$$

with ρ and ω being the given parameters, may be successfully used to terminate the iterative process in eqns (23)–(26). The left-hand side in the inequality in eqn (34) is the largest difference of the two successive approximations of the deflection function. Hence, it does not depend on the value of the parameter τ_{k+2}.

To facilitate our computation, optimal values of the parameters τ_0, α, μ_i and μ_d have been determined in advance by a series of numerical experiments. It has been shown, for instance, that variation in the coefficient of condensing α within the interval 0.1–0.9 does not influence the convergence at all. The optimal values of the coefficient μ_i have been discovered within the interval 1.3–1.4, while those of the parameter μ_d have been found within the interval 0.4–0.6. It has also been shown that the optimal values of the parameter τ_0 belong to the interval 0.005–0.060. Another important detail of the iterative algorithm in eqns (23)–(26) has been discovered. Namely, it appears that the convergence does not depend on the initial approximation. Only the rate of the convergence is slightly affected by the initial approximation.

GENERAL ALGORITHM BASED ON THE GFM

We are now in a position to analyze the geometrically nonlinear bending of the system of N parallel plates. In doing that, we introduce Sobolev's spaces $H_{n2} \equiv [U_2^1(\Omega)]^2$ of vector-functions U_n with components $u_n(x,y), v_n\,(x,y)$ summable over Ω along with their squares and the first order partial derivatives. We construct a direct product $H_2 = H_{12}\otimes H_{22}\otimes \ldots \otimes H_{N2}$ and define a set $T_2 \subset H_2$:

$$T_2 = \{U = \{U_1, U_2, \ldots, U_N\} \in H_2 : \quad C_1^n(u_n, v_n)\,|_{(x,y)\in\Gamma} = 0, \quad C_2^n(u_n, v_n)\,|_{(x,y)\in\Gamma} = 0, \quad n = 1, \ldots, N\}, \tag{35}$$

of kinematically permissible in-plane displacements, including the set of displacements satisfying the boundary conditions in eqn (3). As is shown by Panagiotopoulos (1985), an exact solution U, W of the problem under consideration must satisfy variational relations as written

$$Q_2(W, W^*-W) \geqslant L_2(W^*-W), \quad \forall\, W^* \in T_1, \tag{36}$$

$$Q_3(U, U^*-U) = L_3(U^*-U), \quad \forall\, U^* \in T_2. \tag{37}$$

Here

$$\begin{aligned} Q_2(W, W^*) = Q_1(W, W^*) + \sum_{n=1}^{N} E_n h_n \iint_\Omega &[(1-v_n^2)^{-1} \cdot \{\partial u_n/\partial x + 1/2 \cdot (\partial w_n/\partial x)^2 \\ &+ v_n \cdot [\partial v_n/\partial y + 1/2 \cdot (\partial w_n/\partial y)^2]\} \cdot \partial^2 w_n^*/\partial x^2 + (1-v_n^2)^{-1} \cdot (\partial v_n/\partial y) \\ &+ 1/2 \cdot (\partial w_n/\partial y)^2 + v_n \cdot [\partial u_n/\partial x + 1/2 \cdot (\partial w_n/\partial x)^2)] \cdot \partial^2 w_n^*/\partial y^2 \\ &+ (1+v_n)^{-1} \cdot (\partial u_n/\partial y + \partial v_n/\partial x + \partial w_n/\partial x \cdot \partial w_n/\partial y) \cdot \partial^2 w_n^*/\partial x\,\partial y]\,\mathrm{d}\Omega, \end{aligned}$$

$$\begin{aligned} L_2(W^*) = L_1(W^*), \quad Q_3(U, U^*) = \sum_{n=1}^{N} \iint_\Omega &[\partial u_n/\partial x \cdot \partial u_n^*/\partial x + \partial v_n/\partial y \cdot \partial v_n^*/\partial y \\ &+ v_n \cdot (\partial u_n/\partial x \cdot \partial v_n^*/\partial y + \partial v_n/\partial y \cdot \partial u_n^*/\partial x) + 1/2 \cdot (1-v_n) \cdot (\partial u_n/\partial y + \partial v_n/\partial x) \\ &\times (\partial u_n^*/\partial y + \partial v_n^*/\partial x)]\,\mathrm{d}\Omega, \quad L_3(U^*) = -\sum_{n=1}^{N} \iint_\Omega [(2\partial w_n/\partial x \cdot (\partial^2 w_n/\partial x^2 + (1-v_n) \\ &\times \partial^2 w_n/\partial y^2) - (1+v_n) \cdot \partial w_n/\partial y \cdot \partial^2 w_n/\partial x \partial y) \cdot u_n^* + (2\partial w_n/\partial y \cdot (\partial^2 w_n/\partial y^2 \\ &+ (1-v_n) \cdot \partial^2 w_n/\partial x^2) - (1+v_n) \cdot \partial w_n/\partial x \cdot \partial^2 w_n/\partial x\,\partial y) \cdot v_n^*]\,\mathrm{d}\Omega. \end{aligned}$$

To solve the system in eqns (36) and (37), we apply the method of penalty functions (Lions, 1969). For this purpose, consider a system of variational equations written in the form

$$Q_2(W, W^* - W) - L_2(W^* - W) - \sum_{n=1}^{N} \iint_{\Omega} F_n(W) \cdot (w_n^* - w_n)\,\mathrm{d}\Omega = 0, \quad \forall\, W^* \in T_1^*, \tag{38}$$

$$Q_3(U, U^* - U) - L_3(U^* - U) = 0, \quad \forall\, U^* \in T_2. \tag{39}$$

The nonlinear operators $F_n(W), (n = 1, \ldots, N)$ have been defined earlier in this paper. A solution U^p, W^p of the system above can be considered as an approximate solution for the system in eqns (36) and (37). The inequalities in eqns (16) and (17) can be used again to determine the shapes of the contact zones. The convergence of the sequence $\{U^p, W^p\}$ to the exact solution U, W of the problem under consideration has been discussed by Glowinski *et al.* (1976). As is shown by Washizu (1982), a solution U^p, W^p of the system in eqns (38) and (39), if one exists, satisfies the system of nonlinear differential equations

$$D_1 \cdot \nabla^2\nabla^2 w_1^p - S(u_1^p, v_1^p, w_1^p) - F_1(w_{1p}, w_2^p) = 0, \quad L(u_1^p, v_1^p) - P(w_1^p) = 0, \tag{40}$$

$$D_n \cdot \nabla^2\nabla^2 w_n^p - S(u_n^p, v_n^p, w_n^p) - F_n(w_{n-1}^p, w_n^p, w_{n+1}^p) = 0,$$
$$L(u_n^p, v_n^p) - P(w_n^p) = 0, \quad n = 2, \ldots, N-1, \tag{41}$$

$$D_N \cdot \nabla^2\nabla^2 w_N^p - q - S(u_N^p, v_N^p, w_N^p) - F_N(w_{N-1}^p, w_N^p) = 0, \quad L(u_N^p, v_N^p) - P(w_N^p) = 0 \tag{42}$$

along with the boundary conditions in eqns (2) and (3).

Thus, an approximate solution of the original problem can be interpreted as a solution of the nonlinear boundary value problem in eqns (2), (3) and (40)–(42). In obtaining a numerical solution for this problem, we again take advantage of the non-stationary double-layered iterative scheme discussed by Samarski and Gulin (1989). In doing this, we consider components in the displacement vector for each plate as limits of the functional sequences $\{u_n^k(x, y)\}$, $\{v_n^k(x, y)\}$, and $\{w_n^k(x, y)\}$, $(n = 1, \ldots, N)$ resulting from the sequence of linear boundary value problems written as

$$D_1\nabla^2\nabla^2 w_1^{k+1} = (1 - \tau_{k+1})D_1\nabla^2\nabla^2 w_1^k + \tau_{k+1}[S(u_1^k, v_1^k, w_1^k) + F_1(w_1^k, w_2^k)],$$
$$L(u_1^{k+1}, v_1^{k+1}) = (1 - \tau_{k+1}) \cdot L(u_1^k, v_1^k) + \tau_{k+1}P(w_1^k), \tag{43}$$

$$D_n\nabla^2\nabla^2 w_n^{k+1} = (1 - \tau_{k+1})D_n\nabla^2\nabla^2 w_n^k + \tau_{k+1}[S(u_n^k, v_n^k, w_n^k) + F_n(w_{n-1}^k, w_n^k, w_{n+1}^k)],$$
$$L(u_n^{k+1}, v_n^{k+1}) = (1 - \tau_{k+1}) \cdot L(u_n^k, v_n^k) + \tau_{k+1}P(w_n^k), \tag{44}$$

$$D_N\nabla^2\nabla^2 w_N^{k+1} = (1 - \tau_{k+1})D_N\nabla^2\nabla^2 w_N^k + \tau_{k+1}[S(u_N^k, v_N^k, w_N^k) + F_N(w_{N-1}^k, w_N^k)],$$
$$L(u_N^{k+1}, v_N^{k+1}) = (1 - \tau_{k+1}) \cdot L(u_N^k, v_N^k) + \tau_{k+1}P(w_N^k), \quad n = 2, \ldots, N-1, \tag{45}$$

$$B_1^n(w_n^{k+1})\,|_{(x,y)\in\Gamma} = 0, \quad B_2^n(w_n^{k+1})\,|_{(x,y)\in\Gamma} = 0, \quad n = 1, \ldots, N, \tag{46}$$

$$C_1^n(u_n^{k+1}, v_n^{k+1})\,|_{(x,y)\in\Gamma} = 0, \quad C_2^n(u_n^{k+1}, v_n^{k+1})\,|_{(x,y)\in\Gamma} = 0, \quad n = 1, \ldots, N. \tag{47}$$

To select an effective sequence τ_k of the parameters τ in the iterative scheme above, we utilize the algorithm described earlier. The criterion in eqn (34) has also been used to terminate the iterative process in eqns (43)–(47).

Let $G_L(x, y; \xi, \varsigma)$ represent the Green's matrix of the Lame's system for the region Ω, with the boundary conditions of the form in eqn (3) being prescribed along its contour Γ.

Then a solution to the nonhomogeneous Lame's system $L[U(x,y)] = -\Phi(x,y)$ satisfying the boundary conditions in eqn (3) is expressible in terms of the integral

$$U(x,y) = \iint_{\Omega} G_L(x,y;\xi,\varsigma)\cdot\Phi(\xi,\varsigma)\,d\Omega(\xi,\varsigma). \tag{48}$$

Hence, to successfully run each single loop in the iterative process defined by eqns (43)–(47), it is necessary to accurately compute the proper and improper integrals of the form

$$\iint_{\Omega} K_L(x,y;\xi,\varsigma)\cdot \mathrm{R}_L(\xi,\varsigma)\,d\Omega(\xi,\varsigma), \tag{49}$$

in addition to those integrals of the form in eqns (27) and (28). The kernel matrices $K_L(x,y;\xi,\varsigma)$ in the integral above originate from the effect of the right-hand side operators in eqns (43)–(47) on the Green's matrix $G_L(x,y;\xi,\varsigma)$, while the factor $R_L(\xi,\varsigma)$ is obtained by computing integrals of the form in eqn (49). The numerical procedures for approximating the integrals in eqns (48) and (49) are exactly like those used for the integrals in eqns (27) and (28). Analytical representations for the Green's matrices are also constructed by using the aforementioned technique.

NUMERICAL RESULTS AND DISCUSSION

To illustrate the practical effectiveness of the proposed approach, we first consider the linear formulation for a simply supported square plate with a uniform thickness h, occupying the region Ω $(0 \leqslant x \leqslant a, 0 \leqslant y \leqslant a)$, and situated a distance δ above the Winkler foundation whose coefficient is K. Let the plate be loaded with a uniform transverse pressure q. In this particular case, the system in eqns (18)–(20) degenerates to a single equation:

$$D\cdot\nabla^2\nabla^2 w - F(w) - q = 0, \tag{50}$$

where

$$F(w) = -\gamma(w)\cdot(w-\delta), \quad \varepsilon = K^{-1/2}.$$

The iterative procedure in eqns (23)–(26) was applied to this particular example. For an initial approximation w_0, we used the classical solution of the corresponding linear problem related to the case $F(w) \equiv 0$ (no geometrical constraints are imposed). The criterion in eqn (22) was applied to terminate the iterative process (with $\sigma_1 = 10^{-10}h$ and $\theta_1 = 10^{-4}$). Based on our findings, we recommend that the value of the parameter τ_0 satisfies $1.0 < \tau_0\cdot K < 10.0$.

Using the classical linear plate theory, one can easily verify that if a given value of the uniform transverse loading q does not satisfy the relation

$$q > \delta\cdot D\cdot(0.00406\cdot a^4)^{-1}, \tag{51}$$

then the maximum value of the deflection function does not exceed δ. As a result, the plate does not come into contact with the foundation.

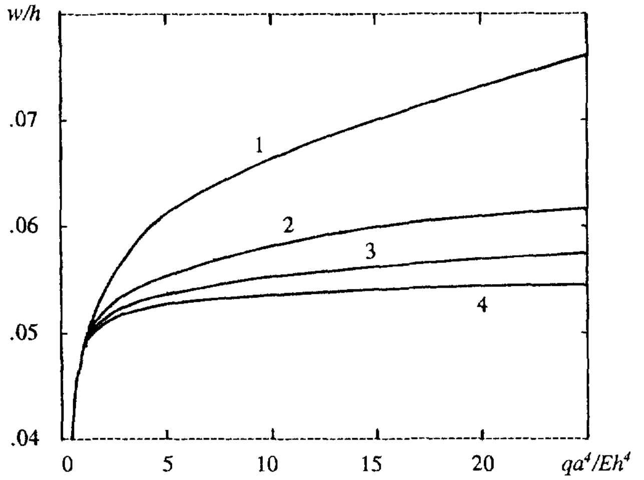

Fig. 2. Square plate contacting the Winkler foundation, geometrically linear case (SPWFL). Deflections vs. loading at the center of the plate.

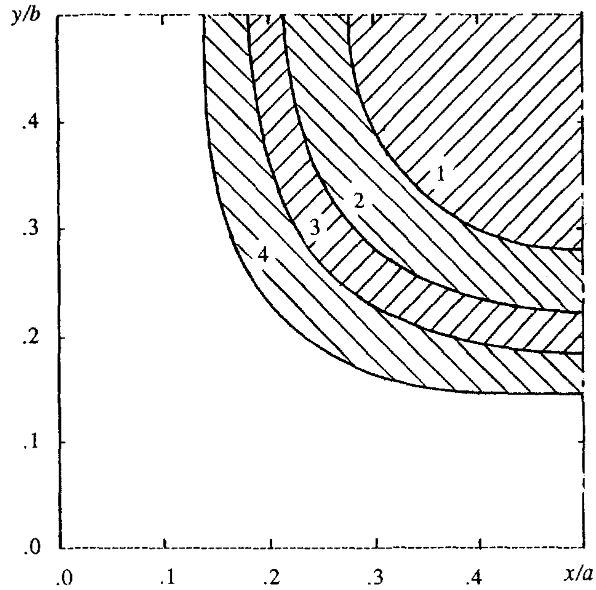

Fig. 3. SPWFL. Shapes of contact zones for different values of the uniform loading q.

Some results from the contact problem specified in eqn (50) are shown in Fig. 2. It shows deflections w_c at the center of the plate as functions of q, for different values of the Winkler coefficient K. Curves 1, 2, 3 and 4 relate to the values K/E equal to 1.25×10^{-5}, 2.5×10^{-5}, 3.75×10^{-5} and 5.0×10^{-5}, respectively. In this formulation, physical and geometrical constants were $E = 0.2 \times 10^6$ MPa, $\nu = 0.3$, $a = 1$ m, $h = 0.01$ m, and $\delta = 0.05\ h$. In order to evaluate integrals in eqn (29), the region Ω was partitioned into $M = 10 \times 10$ of elementary rectangles. Analyzing the information in Fig. 2, one can particularly conclude that for values of K/E exceeding 3.75×10^{-5}, the relative rigidity of the elastic foundation results in practically horizontal shapes of those portions of the q-to-w_c graphs which relate to the values of qa^4/Eh^4 exceeding 5.0.

To determine the shape of the contact zone, the condition in eqn (17) was used. Figure 3 shows the shapes of the contact zones for various values of the uniform loading q. Because of the symmetry in the problem, we exhibit only a quarter of the plate. Figures 4 and 5 show the bending moments M_x and M_y, respectively, along the line $y = a/2$. In Figs 3–5, curves 1, 2, 3, and 4 relate to values of qa^4/Eh^4 equal to 5, 10, 15, and 20, respectively.

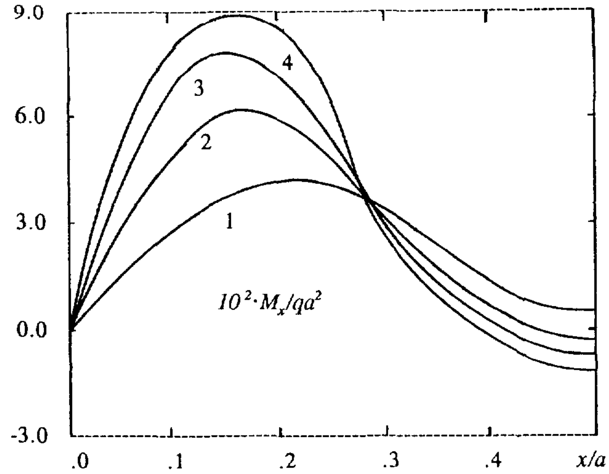

Fig. 4. SPWFL. Distribution of the bending moments M_x along the midline $y = a/2$.

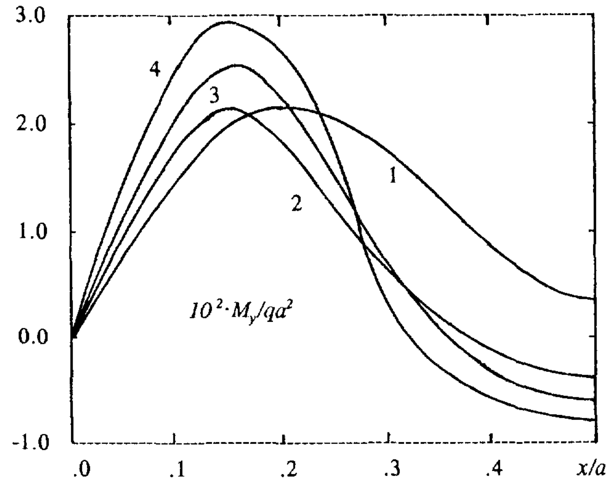

Fig. 5. SPWFL. Distribution of the bending moments M_y along the midline $y = a/2$.

Notice, in particular, that for a relatively low value of the loading q (Fig. 3, curve 1), the shape of the contact zone is close to a circle, while for the greater values of q its shape is notably affected by the shape of the plate. Upon analyzing the results shown in Figs 4 and 5, we realize that the influence of the Winkler foundation causes lower levels of stresses in the central zones of the plates than in the peripheral zones.

To validate the proposed version of the Green's function method, we now state a problem that has an analytic solution for comparison to the GFM solution. Within the scope of the geometrically linear formulation, consider the axisymmetric bending of a simply supported circular plate of radius a. Its bottom surface is situated a distance δ just above the Winkler foundation with coefficient K (see Fig. 6). The deflections are produced by the bending moments M_r uniformly distributed along the outer edge of the plate such that the slope of the deflection surface at that edge is equal to m.

Upon providing the notation $l^4 = D/K$ (Timoshenko and Woinowsky-Krieger, 1959) and introducing the following dimensionless quantities

$$x = r/l, \quad \alpha = a/l, \quad \beta = b/l, \quad z_1 = w_1/l, \quad z_2 = w_2/l,$$

eqn (50) can be reduced to the pair of ordinary differential equations

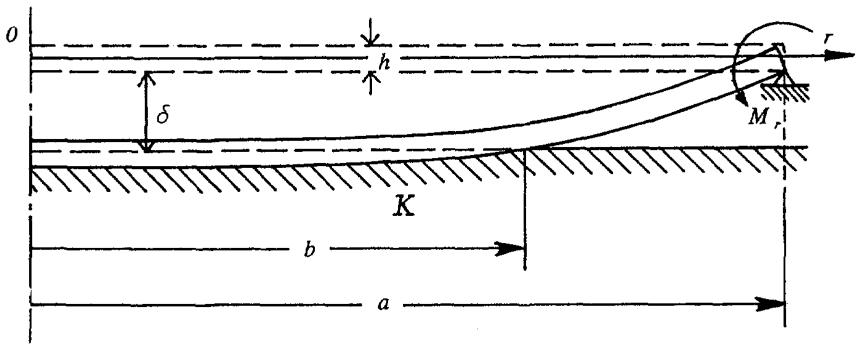

Fig. 6. Axisymmetric bending of the circular plate constrained with the Winkler foundation.

$$(d^2/dx^2+1/x\cdot d/dx)(d^2z_1/dx^2+1/x\cdot dz_1/dx)+z_1=0,\quad 0<x<\beta \tag{52}$$

$$(d^2/dx^2+1/x\cdot d/dx)(d^2z_2/dx^2+1/x\cdot dz_2/dx)=0,\quad \beta<x<\alpha. \tag{53}$$

In this formulation, $w_1 = w_1(r)$ and $w_2 = w_2(r)$ represent the deflection functions of the plate over the intervals $(0, b)$ and (b, a), respectively. In this formulation, l represents the relative rigidity of the foundation. It should be noted that b is an unknown parameter.

According to the statement of the problem, the boundary conditions

$$z_1=z_2=\gamma,\quad dz_1/dx=dz_2/dx,\quad d^2z_1/dx^2+\lambda/x\cdot dz_1/dx=d^2z_2/dx^2+\lambda/x\cdot dz_2/dx,$$
$$d/dx\,(d^2z_1/dx^2+1/x\cdot dz_1/dx)=d/dx\,(d^2z_2/dx^2+1/x\cdot dz_2/dx) \tag{54}$$

are prescribed at $x = \beta$ as those associated with the continuity of the deflections, slopes, bending moments, and shearing forces at that point; $\gamma = \delta/l$ is a dimensionless quantity.

At $x = \alpha$, we have

$$z_2=0,\quad dz_2/dx=\underline{m}. \tag{55}$$

Since the function z_1 and its derivatives at $x = 0$ are bounded, we can express the solutions of eqns (52) and (53) in the form

$$z_1(x)=C_1\,\mathrm{ber}\,(x)+C_2\mathrm{bei}\,(x),\quad z_2=D_1+D_2x^2+D_3\ln(x)+D_4x^2\ln(x), \tag{56}$$

where ber(x) and bei(x) are Bessel functions introduced by Kelvin (Watson, 1948).

The values of C_1, C_2, D_1, D_2, D_3, D_4 as well as that of parameter β (and hence b) can be determined upon satisfying the boundary conditions in eqns (54) and (55). This results in a system of seven equations in seven unknowns. The system is nonlinear with respect to β, but is linear with respect to all the coefficients C_i and D_i. Eliminating these coefficients by using equivalent linear transformations results in the transcendental equation

$$F(\beta)=0. \tag{57}$$

It turns out that $F(\beta)$ is a monotonous function as $\beta \in [0, a]$. Therefore, a numerical solution of eqn (57) can be easily computed with a desired level of accuracy by using any available standard procedure for transcendental equations. This allows one to evaluate afterwards the coefficients C_i and D_i in eqn (56) (and, hence, ultimately obtain the deflections $w_1(r)$, $w_2(r)$, and, consequently, any other component of the stress–strain state).

Tables 1 and 2 list some results indicating the degree of accuracy that can be attained by the GFM approach. Table 1, for instance, shows the values b/a of the relative radius of the contact zone (see Fig. 6) against the values of the slope m, for the following set of initial data: $l = 5.0$, $\alpha = 4.0$, $\gamma = -0.04$, and $\nu = 0.3$. This table also presents the deflection at the center of the plate and radial bending moment at its edge. It can be seen that the relative error for the presented results does not exceed 0.5%. The deflection w and radial bending moment M_r, for the problem determined by $l = 10.0$, $\alpha = 2.5$, $\gamma = -0.1$, $m = 0.015$, and $\nu = 0.3$, are given in Table 2. It appears that for this formulation, the computed value of the relative radius b/a of the contact zone is equal to 1.4836.

It should be especially noted that the bending moment M_r is computed as accurately as the deflection. This positive phenomenon is firmly established by the fact that, within our computation, we absolutely avoid any numerical differentiation. This highlights one of the significant advantages of our version of the GFM. The presented test helps to validate the proposed approach.

Several particular examples are presented below to illustrate the numerical capabilities of the suggested approach as applied to the geometrically nonlinear formulation. The first of them is another validation example, as the problem considered is a single rectangular plate $\{0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b\}$ having a uniform thickness h, and situated a distance δ just above the Winkler foundation whose constant is K. The edges $x = 0$, $y = 0$, and $y = b$ are simply supported, while the edge $x = a$ is clamped. The transverse loading is given by $q(x) = q^* \cdot x/a$. In the present formulation, the physical and geometrical parameters are $E = 0.21 \times 10^6$ Mpa, $\nu = 0.3, a = 1.5$ m, $b = 1.0$ m, $h = 0.5 \times 10^{-2}$ m, $\delta = h$, $K = 0.25 \times 10^{-4} E$. Figures 7–10 exhibit some characteristics of the stress–strain state of the plate. On each of them, curve 1 relates to a value of the transverse loading determined by $q^* = 0.4 \times 10^{-2}$ Mpa, while curve 2 relates to $q^* = 10^{-2}$ MPa. Notice that the first of these cases corresponds with the very beginning of the contact process, while the second case illustrates a latter stage of the process. Figures 7, 8, and 9 show the deflection w, membrane forces N_x, and bending moments M_x, respectively, along the midline $y = b/2$ of the plate. A distribution of the bending moments M_y along the midline $x = a/2$ is seen in Fig. 10. The geometrical constraint notably influences the behavior of the plate, causing a common reduction in the values of all the components of the stress–strain state inside of the contact zone.

For the second example dealing with the geometrically nonlinear formulation, consider the bending of two closely-spaced parallel square $(0 \leqslant x \leqslant a, 0 \leqslant y \leqslant a)$ plates whose edges $x = 0$, a are clamped, while the edges $y = 0$, a are simply supported. Let the upper plate undergo a transverse loading q uniformly distributed over the square Ω^* $(0.4a \leqslant x \leqslant 0.6a, 0.4a \leqslant y \leqslant 0.6a)$. Suppose that the plates have equal thickness h, and the

Table 1. Circular plate contacting the Winkler foundation. The values b/a, $w(0)/h$, and M_r against the slope m

m		0.010	0.012	0.014	0.016	0.018	0.020
b/a	exact	0.2964	0.4410	0.5502	0.6301	0.6882	0.7310
	GFM	0.2957	0.4402	0.5493	0.6294	0.6874	0.7298
$w(0)/h$	exact	0.1066	0.1142	0.1245	0.1372	0.1515	0.1676
	GFM	0.1062	0.1137	0.1240	0.1368	0.1511	0.1671
$10^2 \cdot M_r(a)$	exact	0.0656	0.0920	0.1186	0.1440	0.1681	0.1909
	GFM	0.0653	0.0916	0.1181	0.1434	0.1675	0.1902

Table 2. Deflection $w(r)$ and radial bending moment $M_r(r)$ of the circular plate contacting the Winkler foundation

r/a		0	0.2	0.4	0.6	0.8	1.0
w/h	exact	0.1306	0.1281	0.1191	0.0991	0.0615	0
	GFM	0.1317	0.1274	0.1186	0.0987	0.0607	0
$10^2 \cdot M_r$	exact	0.0247	0.0314	0.0510	0.0820	0.1107	0.1315
	GFM	0.0261	0.0303	0.0504	0.0816	0.1103	0.1297

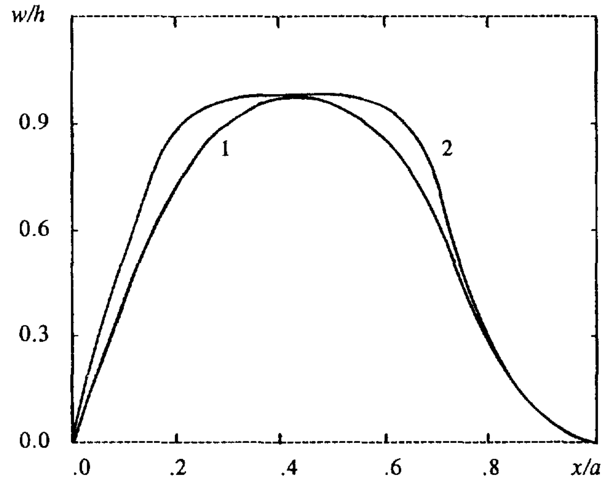

Fig. 7. Rectangular plate contacting the Winkler foundation, geometrically nonlinear case (RPWFN). Deflections w on the midline $y = b/2$ for different loadings.

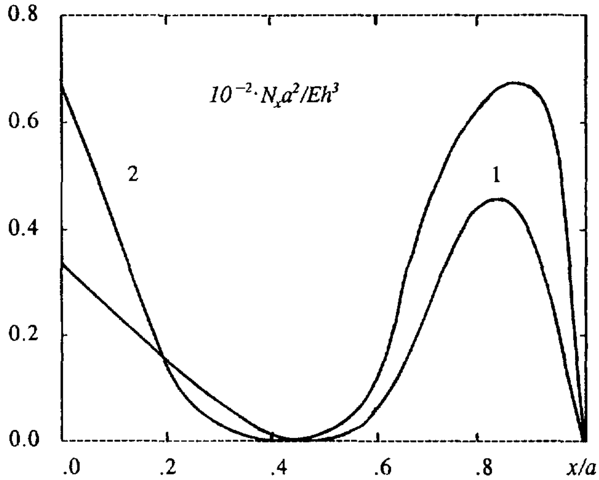

Fig. 8. RPWFN. Distribution of the membrane forces N_x along the midline $y = b/2$.

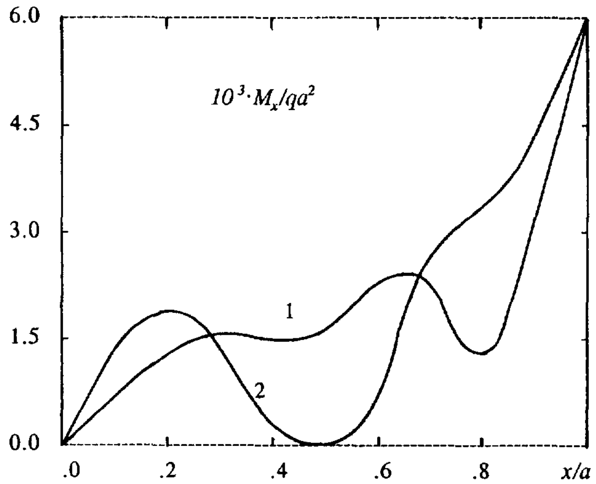

Fig. 9. RPWFN. Distribution of the bending moments M_x along the midline $y = b/2$.

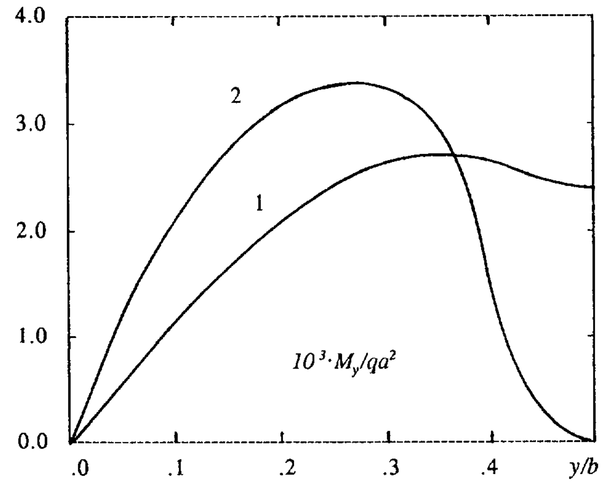

Fig. 10. RPWFN. Distribution of the bending moments M_y along the midline $x = a/2$.

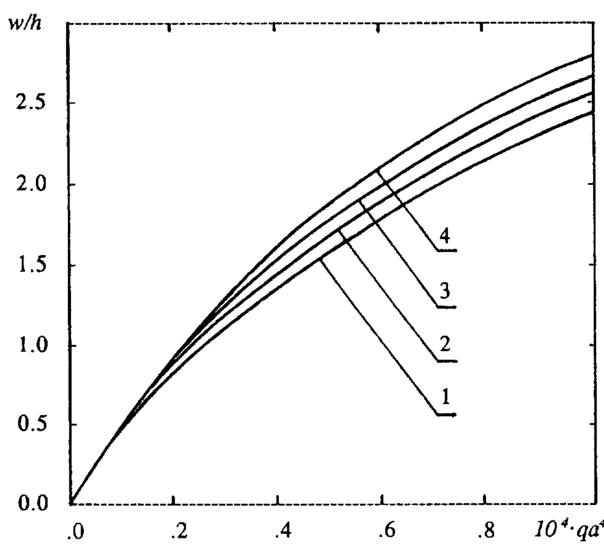

Fig. 11. Two parallel rectangular plates, geometrically nonlinear case (TRPN). Loading-to-deflection dependence at the center of the upper plate.

initial distance between their middle planes is δ. Some numerical results of the solution are presented in Figs 11–16. In this formulation, the physical and geometrical parameters are $E = 0.21 \times 10^6$ MPa, $\nu = 0.3$, $a = 1.0$ m, $h = 0.5 \times 10^{-2}$ m. Figure 11 shows the loading-to-deflection dependence at the center of the upper plate for different values of the initial distance δ. Graphs 1–4 relate to $\delta = 1.5\ h$, $1.75\ h$, $2.0\ h$, and $2.25\ h$, respectively. The nonlinear nature of those graphs does in fact confirm the validity of the nonlinear formulation of the problem for the given loading. Figures 12–14 depict some characteristics of the stress–strain state in the upper plate, while Figs 15 and 16 depict those in the lower plate for $\delta = 1.75\ h$. Graphs 1, 2 and 3 relate to values of the loading q equal to 0.44 MPa, 0.32 MPa, and 0.24 MPa, respectively. Notice that the bending moments in the upper plate are 30–40% higher than those in the lower plate, although their characteristics are identical. It is also important to notice that, similar to the previous problem, the ratios of maximal values of the in-plane displacements to the deflections for both plates do not exceed $10^{-5}-10^{-6}$. These results corroborate the application of linearized contact conditions of the form in eqn (4).

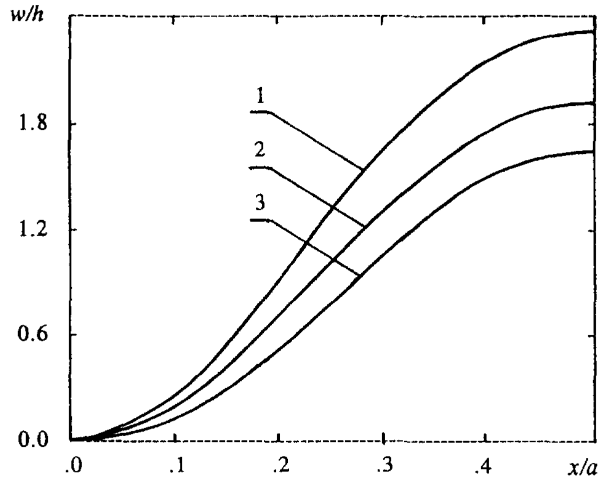

Fig. 12. TRPN. Deflections of the upper plate for different loadings.

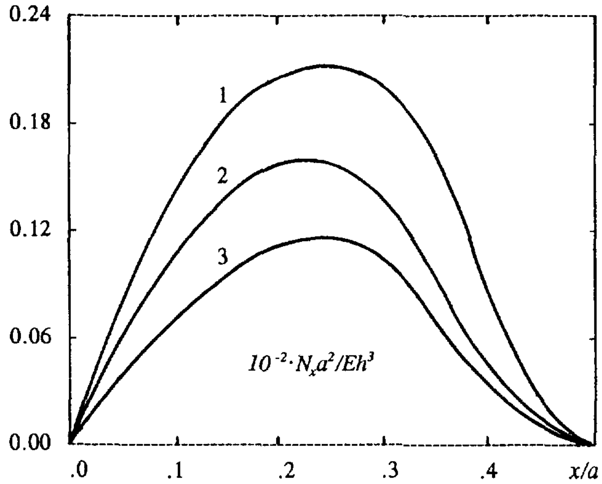

Fig. 13. TRPN. Distribution of the membrane forces N_x for the upper plate.

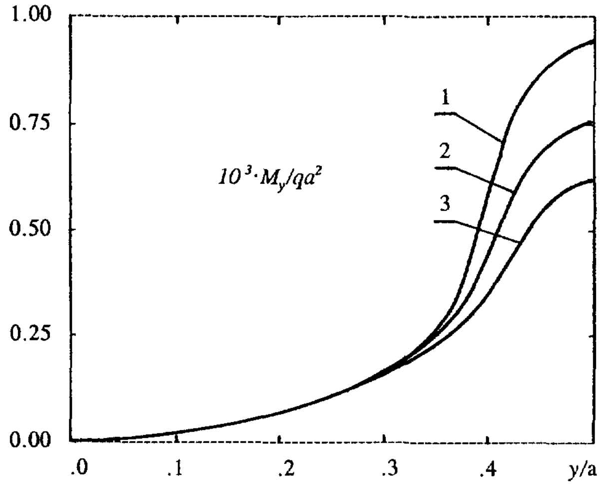

Fig. 14. TRPN. Distribution of the bending moments M_y for the upper plate.

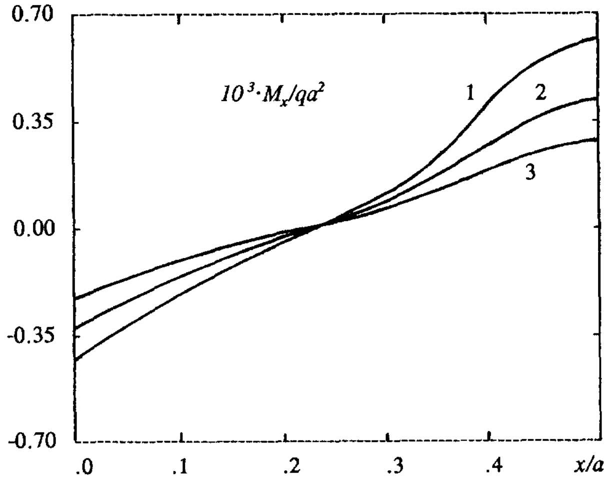

Fig. 15. TRPN. Distribution of the bending moments M_x for the lower plate.

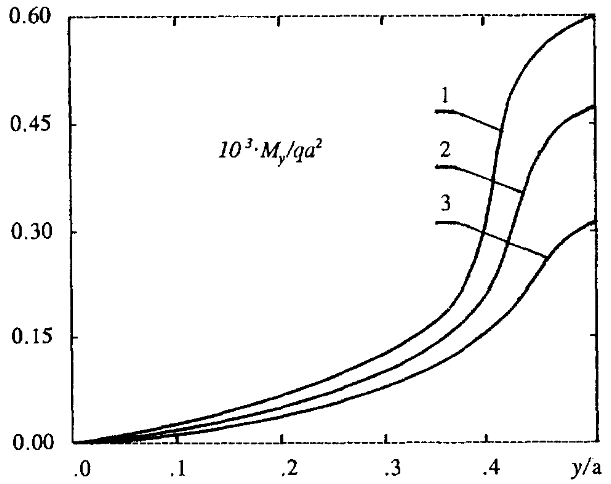

Fig. 16. TRPN. Distribution of the bending moments M_y for the lower plate.

The last example in this paper again involves two parallel, simply supported rectangular plates occupying a region Ω $(0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b)$. The upper plate is undergoing a uniformly distributed transverse loading q, causing the lower plate to contact an elastic foundation whose Winkler coefficient is K. The thicknesses of both the plates are h, the initial distance between their middle planes is $\delta_2 = 1.1\ h$, and the initial distance between the bottom surface of the lower plate and the foundation is $\delta_1 = 0.5\ h$. The physical and geometrical parameters are $E = 0.21 \times 10^6$ MPa, $\nu = 0.3$, $a = 2.0$ m, $b = 1.0$ m, $h = 0.5 \times 10^{-2}$ m, $K = 0.21 \times 10^{-3} E$. Figure 17 depicts shapes of the contact zones for the plates. The shapes of the contact zones for the lower plate and the foundation are seen in Fig. 18. In both figures, curve 1 relates to the value of the loading $q = 10^{-2}$ MPa; curve 2 to $q = 0.3 \times 10^{-2}$ MPa.

For the approximations of the integrals in eqns (27), (28), (48) and (49), we partitioned the region Ω into 10×10 elementary rectangles. In applying the penalty function method, the value $\varepsilon^{-2} = 0.5 \times 10^{-3} E$ of the penalty coefficient had been determined by numerical experiment.

CONCLUDING REMARKS

The complexities of the boundary value problems in this study arise from the simultaneous appearance of two different types of nonlinearities. The first is the geometrical

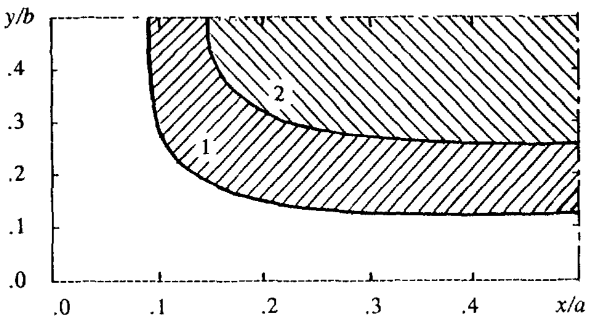

Fig. 17. Two plates above the Winkler foundation, geometrically nonlinear case (TPWFN). Contact zones for the upper and lower plates.

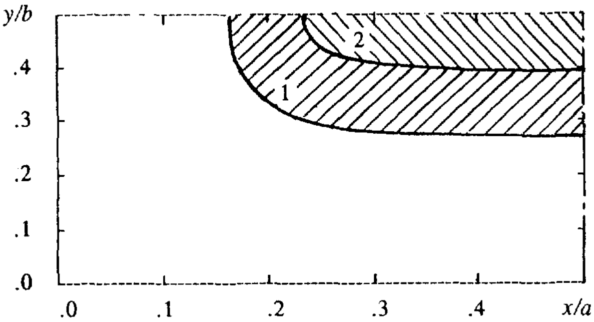

Fig. 18. TPWFN. Contact zones for the lower plate and foundation.

nonlinearity originating from a relatively large level of deflections of the plates. The other results from the fact that the contact zones of neighboring plates are not fixed in advance; thus, the solution process must determine their shapes along with other unknowns. An iterative process which can effectively account for both of these nonlinearities must maintain exceptional accuracy when solving the linear problem within each loop. It is evident that the version of the Green's function method presented in this paper does provide the required level of accuracy.

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